TITLE:
WAVE-FRONT TRACKING FOR THE EQUATIONS OF NON-ISENTROPIC GAS DYNAMICS(Mathematical Analysis in Fluid and Gas Dynamics : A conference in honor of Professor Tai-Ping Liu on his 60th Birthday)

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CITATION:
ASAKURA, FUMIOKI. WAVE-FRONT TRACKING FOR THE EQUATIONS OF NON-ISENTROPIC GAS DYNAMICS(Mathematical Analysis in Fluid and Gas Dynamics : A conference in honor of Professor Tai-Ping Liu on his 60th Birthday). 数理解析研究所講究録 2006, 1495: 78-91

ISSUE DATE:
2006-05

URL:
http://hdl.handle.net/2433/58312

RIGHT:

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1495 巻 2006 年 78-91

WAVE-FRONT TRACKING FOR THE EQUATIONS OF NON-ISENTROPIC GAS DYNAMICS

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Dedicated to Professor Tai-Ping Liu on his 60th birthday

1. INTRODUCTION

The equations of one dimensional gas dynamics in Lagrangian coordinates are given by

$$\left\{\begin{array}{rcl} v_t - u_x &=& 0, \\ u_t + p_x &=& 0, \\ \left(e + \frac{1}{2}u^2\right)_t + (pu)_x &=& 0, \ (x,t) \in \boldsymbol{R} \times \boldsymbol{R}_+, \end{array}\right. \tag{1}$$

Here u is the velocity, p the pressure and v the specific volume satisfying $v > 0$; the thermodynamic quantities satisfy the first and second law: $de = \Theta d\eta - pdv$ (Θ: temperature, e : internal energy). If the gas is *ideal*: $pv = R\Theta$ and *polytropic*: $e = C_v\Theta$, then η is expressed as $\eta = C_v\{\log p + (1 + \frac{R}{C_v})\log v\} + \text{const}$. Hence we have

$$e = \tfrac{pv}{\gamma-1}, \quad p = a^2 v^{-\gamma} e^{\frac{\gamma-1}{R}\eta} \quad (\gamma = 1 + \tfrac{R}{C_v} > 1). \tag{2}$$

If $\gamma = 1$, (1) coincides with the equations of isothermal gas dynamics. We shall discuss the existence of global in time solutions to the initial value problem $(v, u, \eta)|_{t=0} = (v_0(x), u_0(x), \eta_0(x))$. Following results are now theorem classics.

Theorem 1 (Nishida [8]) *Suppose that* $\gamma = 1$. *If the total variation of the initial data* TVv_0, TVu_0, $TV\eta_0$ *are finite, then there exists a global solution.*

Theorem 2 (Liu [6]) *Suppose that* $1 < \gamma \le \frac{5}{3}$. *If* $(\gamma-1)TVv_0$, $(\gamma-1)TVu_0$, $(\gamma-1)TV\eta_0$ *are sufficiently small, then there exists a global solution.*

The above authors have obtained global solutions by using the Glimm difference scheme ([5]). The aim of this article is to to give an alternative proof by using the wave-front tracking scheme (Bressan [2, 3], Risebro [10]).

2. RIEMANN PROBLEM

Note that $v = a^{\frac{2}{\gamma}} e^{\frac{\gamma-1}{R\gamma}\eta} p^{-\frac{1}{\gamma}}$ and $\sqrt{-v_p(p,\eta)} = \gamma^{-\frac{1}{2}} a^{\frac{1}{\gamma}} e^{\frac{\gamma-1}{2R\gamma}\eta} p^{-\frac{\gamma+1}{2\gamma}}$. Quantities p, u, η will be independent variables. Since associated quasi-linear equatations are

$$p_t - \frac{u_x}{v_p} = 0, \quad u_t + p_x = 0, \quad \eta_t = 0,$$

we find by direct computation that the characteristic speeds are

$$\lambda_1(U) = -\frac{1}{\sqrt{-v_p(p,\eta)}}, \quad \lambda_2(U) = \frac{1}{\sqrt{-v_p(p,\eta)}}, \quad \lambda_0(U) = 0 \tag{3}$$

and the corresponding characteristic fields are

$$\begin{gathered} R_1(U) = {}^t(1, \sqrt{-v_p(p,\eta)}, 0), \quad R_2(U) = {}^t(1, -\sqrt{-v_p(p,\eta)}, 0), \\ R_0(U) = {}^t(0,0,1). \end{gathered} \tag{4}$$

Since the integral curves of $R_j(U)$ $(j = 0,1,2)$ are expressed as

$$\begin{array}{ll} R_1(U): & u + \int \sqrt{-v_p}\, dp = \text{const}, \quad \eta = \text{const}, \\ R_0(U): & \qquad p = \text{const}, \quad u = \text{const}, \\ R_2(U): & u - \int \sqrt{-v_p}\, dp = \text{const}, \quad \eta = \text{const}, \end{array} \tag{5}$$

setting $\epsilon = \dfrac{\gamma-1}{2}$, we have rarefaction curves through U_0 in the following form

$$\begin{array}{llcll} \text{1-rarefaction curve:} & u - u_0 & = & -\frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_0}(p^{\frac{\epsilon}{\gamma}} - p_0^{\frac{\epsilon}{\gamma}}), & (p \le p_0), \\ & \eta - \eta_0 & = & 0 & \\ \text{2-rarefaction curve:} & u - u_0 & = & \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_0}(p^{\frac{\epsilon}{\gamma}} - p_0^{\frac{\epsilon}{\gamma}}), & (p \ge p_0). \\ & \eta - \eta_0 & = & 0 & \end{array} \tag{6}$$

A self-similar jump discontinuities having the form

$$U(x,t) = \begin{cases} U_- & \text{for} \quad x < st, \\ U_+ & \text{for} \quad x > st. \end{cases} \tag{7}$$

is a *weak solution,* if and only if it satisfies the *Rankine-Hugoniot condition*:

$$\begin{cases} e - e_0 + \frac{1}{2}(p + p_0)(v - v_0) = 0, \\ (u - u_0)^2 = -(p - p_0)(v - v_0) \end{cases} \tag{8}$$

The shock speed s is expressed as $s^2 = -\dfrac{p - p_0}{v - v_0}$. For the polytropic gas, above condition is equivalent to

$$\begin{cases} e^{\frac{\gamma-1}{R}(\eta-\eta_0)} & = & \left(\dfrac{p}{p_0}\right)\left\{\dfrac{(\gamma-1)p + (\gamma+1)p_0}{(\gamma+1)p + (\gamma-1)p_0}\right\}^{\gamma}, \\ (u - u_0)^2 & = & \dfrac{2v_0(p - p_0)^2}{(\gamma+1)p + (\gamma-1)p_0} \end{cases} \tag{9}$$

In order to solve the Riemann problem, we define the *forward 1-wave curve* $\widehat{\mathcal{W}}_1^F(U_L)$ and the *backward 2-wave curve* $\widehat{\mathcal{W}}_2^B(U_R)$ as the following.

$$
\begin{aligned}
\widehat{\mathcal{W}}_1^F(U_L):\quad & u - u_L = \begin{cases} -\dfrac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_L}\left(p^{\frac{\epsilon}{\gamma}} - p_L^{\frac{\epsilon}{\gamma}}\right) & (p \le p_L) \\ -\dfrac{a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{R\gamma}\eta_L}(p - p_L)}{p_L^{\frac{1}{2\gamma}}\{(1+\epsilon)p + \epsilon p_L\}^{\frac{1}{2}}} & (p > p_L), \end{cases} \\
& \eta - \eta_L = \begin{cases} 0 & (p \le p_L) \\ \dfrac{R}{2\epsilon}\log\left[\left(\dfrac{p}{p_L}\right)\left\{\dfrac{\epsilon p + (1+\epsilon)p_L}{(1+\epsilon)p + \epsilon p_L}\right\}^{\gamma}\right] & (p > p_L), \end{cases} \\
\widehat{\mathcal{W}}_2^B(U_R):\quad & u - u_R = \begin{cases} \dfrac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_R}\left(p^{\frac{\epsilon}{\gamma}} - p_R^{\frac{\epsilon}{\gamma}}\right) & (p \le p_R) \\ \dfrac{a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{R\gamma}\eta_R}(p - p_R)}{p_R^{\frac{1}{2\gamma}}\{(\gamma+1)p + (\gamma-1)p_R\}^{\frac{1}{2}}} & (p > p_R). \end{cases} \\
& \eta - \eta_R = \begin{cases} 0 & (p \le p_R) \\ \dfrac{R}{2\epsilon}\log\left[\left(\dfrac{p}{p_R}\right)\left\{\dfrac{\epsilon p + (1+\epsilon)p_R}{(1+\epsilon)p + \epsilon p_R}\right\}^{\gamma}\right] & (p > p_R). \end{cases}
\end{aligned}
\tag{10}
$$

Each wave curve constitutes a C^2-curve with Lipschitz continuous second derivative. If $(p,u,\eta) \in \widehat{\mathcal{W}}_1^F(U_L)$, then there is a 1-rarefaction wave or shock wave connecting (p_L, u_L, η_L) and (p,u,η). If, on the other hand, $(p,u,\eta) \in \widehat{\mathcal{W}}_2^B(U_R)$, then there is a 2-rarefaction waves or shock wave connecting (p,u,η) and (p_R, u_R, η_R). Let $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$, respectively the projection of $\widehat{\mathcal{W}}_1^F(U_L)$ and $\widehat{\mathcal{W}}_2^B(U_R)$, respectively, onto the pu plane. The Riemann problem is solved as the following way. Let (p_L, u_L, η_L) and (p_R, u_R, η_R) be given Riemann data. If two curves $\mathcal{W}_1^F(U_L)$ and $\mathcal{W}_2^B(U_R)$, have an intersection point (p_m, u_m), then the state $(p_m, u_m, \eta_m^-) \in \widehat{\mathcal{W}}_1^F(U_L)$ and $(p_m, u_m, \eta_m^+) \in \widehat{\mathcal{W}}_2^B(U_L)$ are conected by an entropy wave. Noticing the sound speed is expressed as $c = \sqrt{\gamma p v} = \sqrt{\gamma} a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{R\gamma}\eta} p^{\frac{\epsilon}{\gamma}}$, we have

Theorem 3 *Let $c = \sqrt{\gamma p v}$ be the sound speed. If $u_R - u_L < \frac{2}{\gamma - 1}(c_L + c_R)$, then there is a unique solution to the Riemann problem.*

2. Wave-Front Tracking Scheme

Let h be a positive number. The approximate solutions are constructed in the following way. First, approximate the initial data by a step function $U_0^h(x)$. Let $x_1 < \cdots < x_M$ be the points of discontinuity of $U_0^h(x)$. At each x_m, we solve the Riemann problem setting $U_L = U_0^h(x_m - 0)$, $U_R = U_0^h(x_m + 0)$. If the solution is composed only of shock waves and entropy waves, we adopt this piecewise constant solution itself. If it contains a centred rarefaction wave, we approximate it by several small fans consisting of constant states and jump discontinuities separating them.

Approximate solutions are constructed until neighbouring jump discontinuities interact. If they interact at $t = t_1$, we construct the approximate solution by solving the

Riemann problems with initial data $U^h(x, t_1 - 0)$. We can repeat the above construction as long as the number of jump discontinuities does not diverge within a finite time.

To avoid the breakdown, we introduce a new approximate solution that is called a *simplified Riemann solver* ([3] for details). At each interaction point, the amount of waves generated by the interaction is estimated by the product of the strengths of incoming waves $|\theta_1\theta_2|$. We choose a threshold $\rho > 0$ so that: if $|\theta_1\theta_2| \geq \rho$, then the usual approximate solution is constructed; if $|\theta_1\theta_2| < \rho$, then the new approximate solution is constructed as the following. Suppose that a 2-shock wave β connects U_L and U_M, and a 1-shock wave α connects U_M and U_R. Then we can find two states U'_M, U'_R so that U_L and U'_M are connected by a 1-shock wave with strength $|\alpha|$, and U'_M and U'_R are connected by a 2-shock wave with strength $|\beta|$; the states U_R, U'_R are separated simply by a discontinuous front that propagates with a fixed speed $\widehat{\lambda} > \max|\lambda_j|$. This discontinuous front is called the *non-physical wave*

3. Basic Lemmas

All lemmas and propositions in this section are proved in Liu [6]. We introduce the Riemann invariants corresponding to $\eta_* = \min \eta_0(x)$

$$w = u - \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_*}(p^{\frac{\epsilon}{\gamma}} - 1), \quad z = u + \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_*}(p^{\frac{\epsilon}{\gamma}} - 1) \tag{11}$$

and set

$$\sigma = \frac{z+w}{2} = u, \quad \tau = \frac{z-w}{2} = \frac{\sqrt{\gamma}a^{\frac{1}{\gamma}}}{\epsilon} e^{\frac{\epsilon}{R\gamma}\eta_*}(p^{\frac{\epsilon}{\gamma}} - 1). \tag{12}$$

Strengths of shock waves and rarefaction waves will be measured by w and z. The pressure is expressed as

$$p(\tau) = \left\{1 + \frac{\epsilon\tau}{\sqrt{\gamma}a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{R\gamma}\eta_*}}\right\}^{\frac{\gamma}{\epsilon}}.$$

Since $w + z \pm e^{\frac{\epsilon}{R\gamma}(\eta_0 - \eta_*)}(z - w) = \text{const}$ along rarefaction curves, we find that

$$z - z_0 = \frac{e^{\frac{\epsilon}{R\gamma}(\eta_0-\eta_*)} - 1}{e^{\frac{\epsilon}{R\gamma}(\eta_0-\eta_*)} + 1}(w - w_0) : \quad \text{1-rarefaction curve,}$$

$$w - w_0 = \frac{e^{\frac{\epsilon}{R\gamma}(\eta_0-\eta_*)} - 1}{e^{\frac{\epsilon}{R\gamma}(\eta_0-\eta_*)} + 1}(z - z_0) : \quad \text{2-rarefaction curve.}$$

Note that $\left|\frac{e^{\frac{\epsilon}{R\gamma}(\eta_0-\eta_*)} - 1}{e^{\frac{\epsilon}{R\gamma}(\eta_0-\eta_*)} + 1}\right| < 1$. We always assume that all waves considered will be in the region

$$0 < \underline{p} \leq p \leq \overline{p}, \quad \epsilon(\eta - \eta_*) \leq K.$$

Proposition 1 *Suppose that*

$$(p_2, u_2, \eta_2) \in \widehat{S}_1^F(p_1, u_1, \eta_1), \quad (p_1, u_1, \eta_3) \in \widehat{\mathcal{R}}_1^F(p_3, u_3, \eta_3)$$
$$\textit{and} \quad (p_3, u_3, \eta_4) \in \widehat{\mathcal{R}}_2^F(p_2, u_2, \eta_4).$$

Then there exists a constant D_* such that $0 < D_* < 1$ and

$$z_3 - z_2 \le D_*(w_1 - w_2), \tag{13}$$

where D_* depends only on $\underline{p}, \overline{p}$ and K.

We set

$$\begin{aligned} g(\tau_0, \tau) &= \frac{a^{\frac{1}{\gamma}} e^{\frac{\epsilon}{R\gamma}\eta_0}(p - p_0)}{p_0^{\frac{1}{2\gamma}} \{(1+\epsilon)p + \epsilon p_0\}^{\frac{1}{2}}} & (14) \\ h(\tau_0, \tau) &= \frac{R}{2\epsilon} \log \left[\left(\frac{p}{p_0} \right) \left\{ \frac{\epsilon p + (1+\epsilon)p_0}{(1+\epsilon)p + \epsilon p_0} \right\}^{\gamma} \right], & (15) \end{aligned}$$

where $p = p(\tau)$ and $p_0 = p(\tau_0)$. By using these function, the Hugoniot curves through (p_0, u_0, η_0) are expressed as

$$\sigma - \sigma_0 = \mp g(\tau_0, \tau), \quad \eta - \eta_0 = h(\tau_0, \tau).$$

We can define the forward 1-shock curve and the backward 2-shock curves with initial state $U_0 = (p_0, u_0, \eta_0)$ denoted by $\widehat{\mathcal{S}}_1^F(U_0)$ and $\widehat{\mathcal{S}}_2^B(U_0)$, respectively. We consider there projection onto pu plane, $\mathcal{S}_1^F(U_0)$ and $\mathcal{S}_2^B(U_0)$, respectively, using the Riemann invariant coordinates.

Lemma 1 *If $0 < \epsilon \le \frac{1}{3}$, then there exist functions $z = z_1(w)$ and $w = w_2(z)$ such that*

$$\begin{aligned} \widehat{\mathcal{S}}_1^F(U_0) &= \{(w, z, \eta);\ z = z_1(w),\ \eta = \eta(\tau),\ w < w_0\}, \\ \widehat{\mathcal{S}}_2^B(U_0) &= \{(w, z, \eta);\ w = w_2(z),\ \eta = \eta(\tau),\ z > z_0\}. \end{aligned}$$

Moreover there exist constants $0 < C_0 < 1$ and $C_1 > 0$ such that

$$0 = z_1(w_0) = w_2(w_0) < z_1'(w), \quad w_2'(z) < C_0, \tag{16}$$

$$z_1''(w) < 0 < w_2''(z), \tag{17}$$

$$0 = \eta'(\tau_0) < \eta'(\tau) < C_1. \tag{18}$$

Recall that $0 < \epsilon \le \frac{1}{3}$ is equivalent to $1 < \gamma \le \frac{5}{3}$. This lemma follows from

Proposition 2 *If $0 < \epsilon \le \frac{1}{3}$, then it follows*

$$g_{\tau\tau}(\tau, \tau_0) > 0 \quad for \quad p > p_0. \tag{19}$$

Lemma 2 *Suppose that*

$$(p_2, u_2, \eta_2) \in \widehat{\mathcal{S}}_1^F(p_1, u_1, \eta_0), \quad (p_4, u_4, \eta_3) \in \widehat{\mathcal{S}}_1^F(p_3, u_3, \eta_0), \quad \tau_3 > \tau_1$$
$$and \quad u_1 - u_2 = u_3 - u_4.$$

Then there exists a constant C_2 such that

$$0 < (\tau_4 - \tau_3) - (\tau_2 - \tau_1) \le \left\{ \begin{array}{l} C_2\epsilon(\tau_3 - \tau_1)(\tau_2 - \tau_1), \\ C_2\epsilon(\tau_3 - \tau_1)(\tau_4 - \tau_3). \end{array} \right. \tag{20}$$

This lemma follows from the observation that $g_{\tau\tau_0} + g_{\tau\tau} < 0$ and the constant C_2 is defined by

$$C_2 = \sup_{\substack{0<\epsilon\leq 1/3 \\ \underline{p}\leq p\leq \overline{p}}} \left| \frac{g_{\tau\tau_0} + g_{\tau\tau}}{\epsilon} \right|$$

Lemma 3 *Suppose that*

$$(p_1, u_1, \eta_1),\ (p_2, u_2, \eta_2) \in \widehat{S}_1^F(p_0, u_0, \eta_0), \quad p_2 > p_1,$$
$$(p_3, u_3, \eta_3) \in \widehat{S}_1^F(p_1, u_1, \eta_0), \quad \textit{and} \quad \tau_2 = \tau_3$$

Then there exists constants C_2 and C_3 such that

$$0 < u_2 - u_3 \leq C_3\epsilon(\tau_1 - \tau_0)(\tau_2 - \tau_1), \tag{21}$$

$$0 < (\eta_2 - \eta_1) - (\eta_3 - \eta_0) \leq C_3\epsilon(\tau_1 - \tau_0)(\tau_2 - \tau_1), \tag{22}$$

Since $g_{\tau\tau} > 0$, we have $g_{\tau\tau_0} < 0$. C_3 is defined by

$$C_3 = \sup_{\substack{0<\epsilon\leq 1/3 \\ \underline{p}\leq p\leq \overline{p}}} |g_{\tau\tau_0}|$$

Lemma 4 *Suppose that*

$$(p_2, u_2, \eta_2) \in \widehat{S}_1^F(p_1, u_1, \eta_1), \quad (p_4, u_4, \eta_4) \in \widehat{S}_1^F(p_3, u_3, \eta_3),$$
$$\textit{and} \quad \tau_1 - \tau_2 = \tau_3 - \tau_4.$$

Then there exists constants C_4 and C_5 such that

$$|(\eta_3 - \eta_1) - (\eta_4 - \eta_2)| \leq C_4\epsilon(\tau_2 - \tau_1)|\tau_1 - \tau_3|, \tag{23}$$

$$|(u_1 - u_2) - (u_3 - u_4)| \leq C_5\epsilon(\tau_2 - \tau_1)(|\eta_1 - \eta_3| + |\tau_1 - \tau_3|). \tag{24}$$

Moreover: if $\tau_1 = \tau_3$ and $\eta_1 > \eta_3$, then

$$0 < (u_1 - u_2) - (u_3 - u_4) \leq C_6\epsilon(\tau_2 - \tau_1)(\eta_1 - \eta_3); \tag{25}$$

if instead $u_2 = u_4$ and $\eta_1 > \eta_3$, then

$$0 < \tau_4 - \tau_2 \leq C_6\epsilon(\tau_2 - \tau_1)(\eta_1 - \eta_3). \tag{26}$$

We observe that $h_{\tau\tau} + h_{\tau\tau_0} < 0$ and $g_{\tau\eta_0} > 0$. Constants are defined by

$$C_5 = \sup_{\substack{0<\epsilon\leq 1/3 \\ \underline{p}\leq p\leq \overline{p}}} \left| \frac{h_{\tau\tau_0} + h_{\tau\tau}}{\epsilon} \right| \quad \text{and} \quad C_6 = \sup_{\substack{0<\epsilon\leq 1/3 \\ \underline{p}\leq p\leq \overline{p}}} \frac{g_{\tau\eta_0}}{\epsilon}$$

Lemma 5 *Suppose that*

$$(p_1,u_1,\eta_1),\ (p_2,u_2,\eta_2) \in \widehat{\mathcal{S}}_1^F(p_0,u_0,\eta_0), \quad p_2 > p_1,$$
$$(p_3,u_3,\eta_3) \in \widehat{\mathcal{S}}_1^F(p_1,u_1,\eta_1), \quad \textit{and} \quad u_3 = u_2.$$

Then there exists a constant C_7 such that

$$\eta_2 - \eta_3 + C_7\epsilon(\tau_1-\tau_0)(\tau_2-\tau_1) \geq 0. \tag{27}$$

If $\gamma = 1$, we have

$$\eta - \eta_0 = \sinh\frac{\tau-\tau_0}{a} - (\tau_1-\tau_0) \quad (\tau = a\log p).$$

Hence we find by direct computation

$$\begin{aligned} \eta_2 - \eta_3 &= \sinh\frac{\tau_2-\tau_0}{a} - \sinh\frac{\tau_1-\tau_0}{a} - \sinh\frac{\tau_3-\tau_2}{a} + \tau_3 - \tau_2 \\ &\geq \tau_3-\tau_2 \geq 0. \end{aligned}$$

This lemma says that the quantity $\eta_2 - \eta_3$ is different from that in the case $\gamma = 1$ by $O(1)\epsilon(\tau_1-\tau_0)(\tau_2-\tau_1)$.

4. Interaction of Two Incoming Waves

We denote by $\alpha, \beta, \delta, \xi, \pi$, respectively, strengths of 1-shock wave, 2- shock wave, 1-rarefaction wave, 2-rarefaction wave, respectively. They are defined as follows

$$\begin{aligned} &\alpha = w_0 - w && \text{if} \quad (p_1,u_1,\eta_1) \in \widehat{\mathcal{S}}_1^F(p_0,u_0,\eta_0), \\ &\beta = z - z_0 && \text{if} \quad (p_1,u_1,\eta_1) \in \widehat{\mathcal{S}}_2^B(p_0,u_0,\eta_0), \\ &\delta = \eta_1 - \eta_0 && \text{if} \quad (p_0,u_0,\eta_1), (p_0,u_0,\eta_0) \\ & && \qquad \text{constitute an entropy wave}, \\ &\xi = w - w_0 && \text{if} \quad (p_1,u_1,\eta_0) \in \widehat{\mathcal{R}}_1(p_0,u_0,\eta_0), \\ &\pi = z_0 - z && \text{if} \quad (p_1,u_1,\eta_0) \in \widehat{\mathcal{R}}_2(p_0,u_0,\eta_0). \end{aligned}$$

In order to measure the increase of the entropy across shock waves, we introduce the quantities $\delta_\alpha, \delta_\beta$ as follows

$$\begin{aligned} &\delta_\alpha = \eta_1 - \eta_0 \quad \text{if} \quad (p_1,u_1,\eta_1) \in \widehat{\mathcal{S}}_1^F(p_0,u_0,\eta_0), \\ &\delta_\beta = \eta_1 - \eta_0 \quad \text{if} \quad (p_1,u_1,\eta_1) \in \widehat{\mathcal{S}}_2^B(p_0,u_0,\eta_0). \end{aligned}$$

¿From now on, we also denote by $\alpha, \beta, \delta, \xi, \pi$, the corresponding waves themselves. Suppose that U_L and U_m are connected by a 2-wave θ_1(or an entropy wave), and U_m and U_R a 1-wave θ_2(or an entropy wave); these two waves are assumed to be incoming and interact. There exists a unique solution to the Riemann problem connecting the states U_L and U_R. This solution is composed of a 1-wave θ' connecting the states U_L

and U_M^-, an entropy wave δ' connecting U_M^- and U_M^+, and 2-wave θ_2' connecting U_M^+ and U_R, which are outgoing waves. This interaction is simply denoted by

$$\theta_2 + \theta_1 \to \theta_1' + \delta' + \theta_2'.$$

Note that

$$\delta' = \delta + \delta_{\alpha'} - \delta_\alpha + \delta_{\beta'} - \delta_\beta. \tag{28}$$

Local interaction estimates are carried out in the same way as [6, 7]. For important cases, we have

Lemma 6 *There exist constants $0 < D_0 < 1$ and $D > 0$ such that the following estimates hold*

(1) $\beta + \alpha \to \alpha' + \delta' + \beta'$:

$$\begin{array}{rcl} \alpha' & \le & \alpha + \epsilon D\alpha\beta, \qquad \delta_{\alpha'} \ \ge \ \delta_\alpha - \epsilon D\alpha\beta, \\ \beta' & \le & \beta + \epsilon D\alpha\beta, \qquad \delta_{\beta'} \ \ge \ \delta_\beta - \epsilon D\alpha\beta, \\ |\delta'| & \le & \epsilon D\alpha\beta. \end{array}$$

(2) $\xi + \alpha \to \alpha' + \delta' + \beta'$:

$$\begin{array}{rcl} \alpha' & \le & \alpha - \xi, \qquad \delta_{\alpha'} \ \ge \ \delta_\alpha - D(\alpha - \alpha') - \epsilon D\alpha\xi, \\ \beta' & \le & D_0(\alpha - \alpha') + \epsilon D\alpha'\xi, \qquad \delta_{\beta'} \ \ge \ 0, \\ |\delta'| & \le & D(\alpha - \alpha') + \epsilon D\alpha\xi. \end{array}$$

(3) $\alpha_1 + \alpha_2 \to \alpha' + \delta' + \pi'$:

$$\begin{array}{rcl} \alpha' & \le & \alpha_1 + \alpha_2, \qquad \delta_{\alpha'} \ \ge \ \delta_{\alpha_1} + \delta_{\alpha_2} - \epsilon D\alpha\xi, \\ \pi' & \le & D\alpha_1\alpha_2, \qquad \delta_{\beta'} \ \ge \ 0, \\ |\delta'| & \le & D\alpha_1\alpha_2. \end{array}$$

(4) $\delta + \xi \to \xi' + \delta' + \theta'$ ($\theta = \beta$ *or* π):

$$\begin{array}{rcl} \xi' & \le & \xi + \epsilon D\delta\xi, \\ |\delta'| & \le & |\delta| + \epsilon D\delta\xi, \\ \theta' & \le & \epsilon D\delta\xi. \end{array}$$

Remark 1 *It is worth noticing that the interaction estimates in the case: $\beta + \alpha \to \alpha' + \delta' + \beta'$ has the form simpler than those of isentropic gas dynamics (see Nishida-Smoller [9]).*

In the above cases, waves of quadratic amplitudes are called *scattered* waves generated through the interaction. Note that they have the estimates $\epsilon D\alpha\beta, \epsilon D\delta\xi$ in (1) and (4), and $D\alpha_1\alpha_2$ in (3). Since β' and δ' in (2) are not scattered waves, these waves will be studied in a different manner. Let M_0 be a small number such that $D_0 + DM_0 < 1$.

The strength of the entropy will be measured by $M_0\delta$. Next section we will find that $\epsilon(\alpha+\xi)$ will be so small that

$$D_0 + \{M_0 + \epsilon(\alpha+\xi)/M_0\}D = D_1 < 1.$$

Hence we have for Case (2)

$$\beta' + M_0|\delta'| \le D_1(\alpha - \alpha'). \tag{29}$$

These estimates will be used in Section 6.

5. Global Interaction Estimates

For space-like curve J, we define the global interaction potential as the following way.

$$\begin{aligned}
F(J) &= L(J) + \epsilon Q(J),\\
L(J) &= \sum_J \{(\alpha - M_0\delta_\alpha) + (\beta - M_0\delta_\beta) + M_0|\delta|\}\\
Q(J) &= M_0M_1\sum_J(\alpha+\beta+\xi+\pi)|\delta|\\
&\quad + M_1\sum_J(\xi\alpha+\xi\beta+\pi\alpha+\pi\beta) + M_2\sum_J(\alpha_1\alpha_2+\alpha\beta+\beta_1\beta_2).
\end{aligned}$$

Here $\sum_J$ denotes the summation of waves crossing J. Let P denote a point of interaction of two waves θ_1, θ_2. We simply define $Q(\mathrm{P}) = |\theta_1\theta_2|$. As [6], we have

Lemma 7 *Suppose that $0 < \epsilon \le \frac{1}{3}$, and ϵTVp_0, ϵTVu_0 and $\epsilon TV\eta_0$ are sufficiently small. Suppose also that constants M_0, M_1, M_2 are chosen so that $M_0 << 1, M_0M_1 = 4, M_2 >> 1$. Then there exists a constant c such that that $F(J') - F(J) \le -c\epsilon Q(\mathrm{P})$ for any space-like curves satisfying $J < J'$ and $U^h(J), U^h(J')$ are contained in the region $0 < \underline{p} \le p \le \overline{p}$, $\epsilon(\eta - \eta_*) \le K$.*

The above lemma implies the following important estimate

$$\epsilon\sum_{\mathrm{P}} Q(\mathrm{P}) \le \frac{F(O)}{c} \tag{30}$$

showing that ϵ times total amount of interaction is uniformly bounded.

6. Decomposition by Paths

Let us consider an approximate solution $U^h(x,t)$ for $0 \le t < T$. A sequence of interaction points $\mathrm{P}_0, \mathrm{P}_1, \ldots, \mathrm{P}_n$ constitutes a *main path*, if $\mathrm{P}_0 \in \{t = 0\}$ and each segment $\mathrm{P}_{j-1}\mathrm{P}_j$ is a shock front or an entropy wave that is *not* a scattered wave; this main path is denoted by

$$\Gamma:\ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n.$$

As Temple-Young [13] (see also Asakura [1]), we define the *index* (c_j, k_j) of each segment $\mathrm{P}_{j-1}\mathrm{P}_j$ in the following way: by setting $k_1 = 1$

$$c_j = \begin{cases} 1, & \text{if } \mathrm{P}_{j-1}\mathrm{P}_j \text{ is a 1-shock wave} \\ 2, & \text{if } \mathrm{P}_{j-1}\mathrm{P}_j \text{ is a 2-shock wave,} \\ 0, & \text{if } \mathrm{P}_{j-1}\mathrm{P}_j \text{ is an entropy wave} \end{cases}$$

$$k_j = \begin{cases} k_{j-1}, & \text{if } c_j = c_{j-1} \\ k_{j-1} + 1, & \text{if } c_j \neq c_{j-1}. \end{cases}$$

Each k_j is called the *generation order* of the segment and the sequence (c_1, k_1), (c_2, k_2), $\ldots$, (c_n, k_n) the *index of the main path.* We observe by Lemma 6 that once the index becomes $(0, k_j)$, then it does not change.

In the same manner as [1], we define the *strength* of the path by using the property (29) and denote by $\alpha_\Gamma(t)$ the strength of Γ^m at t. Each wave $\mathrm{P}_{j-1}\mathrm{P}_j$, that is different from a scattered wave, is contained in a finite number of main paths and the amplitude is the sum of the amplitudes of these paths. When a scattered shock wave or entropy wave, denoted by $\mathrm{P}_{j_0-1}\mathrm{P}_{j_0}$, is generated, a *branch path* starts from P_{j_0-1}. The generation number of the branch path is defined to be 1+ the maximum of those of incoming waves. If a branch shock or entropy wave interacts with a wave, all generated waves whose characteristic directions are different from those of incoming waves are considered to be new scattered waves. Hence the index of the branch path does not change. A path Γ is considered to be a Lipschitz curve $x = \Gamma(t)$. A collection of a finite number of main paths $\mathbf{\Gamma^m} = \{\Gamma^m_j\}$ and that of $\mathbf{\Gamma^b} = \{\Gamma^b_j\}$ are defined in the approximate solution. The generation order of Γ at t is denoted by $k_\Gamma(t)$. We have

Lemma 8 *For every approximate solution, we have a collection of a finite number of main paths* $\mathbf{\Gamma^m} = \{\Gamma^m_j\}$ *and branch paths* $\mathbf{\Gamma^b} = \{\Gamma^b_j\}$ *such that*

1. $L^-(t) = \sum_{\Gamma \in \mathbf{\Gamma^m} \cup \mathbf{\Gamma^b}} \alpha_\Gamma(t)$

2. *Let* $\Gamma^m : \ \mathrm{P}_0 \to \mathrm{P}_1 \to \cdots \to \mathrm{P}_n$ *be a main path, and* $(c_j, k_j), \alpha_j$ *its index and strength, respectively, of* $\mathrm{P}_{j-1}\mathrm{P}_j$. *Then there exists* $0 < \chi < 1$ *and*

$$\begin{aligned} k_{j+1} = k_j \quad &\Rightarrow \quad \alpha_{j+1} \leq (1 + C\epsilon|\beta_j|)\alpha_j, \\ k_{j+1} = k_j + 1 \quad &\Rightarrow \quad \alpha_{j+1} \leq \chi\alpha_j \end{aligned}$$

3. *Let* $\Gamma^s : \ \mathrm{P}_{j_0} \to \mathrm{P}_{j_0+1} \to \cdots \to \mathrm{P}_n$ *be a branch path and* α_j *its strength of* $\mathrm{P}_{j-1}\mathrm{P}_j$. *Then the indices are constant and*

$$\alpha_{j+1} \leq (1 + C\epsilon|\beta_j|)\alpha_j.$$

In the both cases, β_j *denotes the interacting wave.*

Lemma 9 *Let $\Gamma = \Gamma^m$ be a main path. Assume that $\epsilon F(O)$ is sufficiently small. Then there exists positive constant κ depending only on χ and satisfying $0 < \kappa < 1$ such that*

$$\begin{aligned} &\alpha_\Gamma(t) \le 2\alpha_\Gamma(0) && if \quad k_\Gamma(t) = 1, \\ &\alpha_\Gamma(t) \le \kappa^{j-1}\alpha_\Gamma(0) && if \quad k_\Gamma(t) = j \ge 2. \end{aligned} \tag{31}$$

Using this lemma, we have

$$\sum_{\Gamma^m : k_\Gamma(t)=j} \alpha_\Gamma(t) \le \kappa^{j-1} \sum_{\Gamma^m : k_\Gamma(t)=1} \alpha_\Gamma(0) = \kappa^{j-1} L^-(0).$$

Denoting $L_j^m(t) = \sum_{\Gamma^m : k_\Gamma(t)=j} \alpha_\Gamma(t)$, we obtain

Proposition 3 *Assume that $\epsilon F(O)$ is sufficiently small. Then there exists positive constant κ depending only on δ and satisfying $0 < \kappa < 1$ such that*

$$L_1^m(t) \le 2L^-(0), \quad L_j^m(t) \le \kappa^{j-1} L^-(0) \ (j \ge 2). \tag{32}$$

Let $L_j^s(t) = \sum_{\Gamma^s : k_\Gamma(t)=j} \alpha_\Gamma(t)$ be the total amount of scattered waves whose generation order is j and $Q_j(\mathrm{P})$ the amount of scattered waves whose generation order is j. It follows from the estimate (30) that

$$\sum_{j \ge 1} \sum_{\mathrm{P}} Q_j(\mathrm{P}) = \sum_{\mathrm{P}} Q(\mathrm{P}) \le \frac{F(O)}{c\epsilon}. \tag{33}$$

Hence denoting $L_j^s = \sum_{\mathrm{P}} Q_j(\mathrm{P})$, we have

Proposition 4 *Assume that $\epsilon F(O)$ is sufficiently small. Then there exists positive sequence $\{L_j^s\}$ such that*

$$L_j^s(t) \le L_j^s \quad and \quad \sum_{j \ge 1} L_j^s < \infty. \tag{34}$$

These propositions will provide estimates of the total amount of non-physical waves generated by the interaction of waves whose generation orders are larger than k and the threshold parameter will be chosen according to the above estimate.

6. Stability of Wave-Front Tracking Scheme

First we prove that the approximate solution is constructed for all $0 \le t < \infty$. Let us assume the contrary. Suppose that there is a sequence of interaction time T_m such that $\lim_{m \to \infty} T_m = T_\infty < \infty$. Since the estimates (30) is true for $0 < t < T_\infty$, there exists a uniform constant C_∞ such that

$$\sum_{0 < t_m < T_\infty} Q(\mathrm{P}_m) \le C_\infty \tag{35}$$

where t_m denotes the interaction time at P_m and the summation runs over the all interaction points between $t = 0$ and T_∞. Let ρ be a threshold introduced in Section 2. The above estimate says that there are less than C_∞/ρ interaction points such that the strengths of incoming waves satisfy $Q(\mathrm{P}_m) \geq \rho$: Since new *physical* fronts are generated only at such points, the number of physical fronts is thus finite. A new non-physical front is generated through the interaction of two physical fronts and any two physical fronts can interact only once. Hence the number of non-physical fronts is also finite. Consequently, we conclude that total number of fronts is finite; this is the contradiction.

Let α be a main shock or entropy wave at t containing several paths which can be arranged so that

$$\Gamma_1(t),\ \Gamma_2(t),\ \Gamma_3(t),\ \ldots \quad (k_{\Gamma_1}(t) \leq k_{\Gamma_2}(t) \leq k_{\Gamma_3}(t) \leq \cdots). \tag{36}$$

The *generation order* of α is defined by $k_{\Gamma_1}(t)$ and denoted by k_α that accords with the definition of Bressan [3]. Let $V_j^m(t)$ be the total amount of main shock or entropy waves at t whose generation orders are larger than j. Then it follows that $V_j^m(t) = \sum_{l\geq j} L_l^m(t)$ and from Proposition 3

$$\sup_{t\geq 0} V_j^m(t) \leq L^-(0) \sum_{l\geq j} \kappa^{l-1} = \frac{\kappa^{j-1} L^-(0)}{1-\kappa}. \tag{37}$$

In the same manner, we define $V_j^s(t) = \sum_{l\geq j} L_l^s(t)$ that is the total amount of branch shock or entropy waves at t whose generation orders are larger than j. Setting $\mu_j = \sum_{l\geq j} L_j^s$, we find that

$$\sup_{t\geq 0} V_j^s(t) \leq \mu_j \to 0 \quad \text{as} \quad j \to \infty. \tag{38}$$

Now we shall carry out the estimates of non-physical waves. Note that only the simplified Riemann solver generates a non-physical wave and non-physical waves do not interact each other. Let ϵ denote an arbitrary non-physical wave. We have the following estimates.

$$(1)\ |\epsilon| \leq D\rho, \quad (2) \sum_{\substack{\epsilon\in\mathcal{NP} \\ k_\epsilon \geq j}} |\epsilon| \leq C_0 \sup_{t\geq 0}(V_j^m(t) + V_j^s(t)). \tag{39}$$

The first estimate comes from Lemma 6. Since

$$\sum_{\substack{\epsilon\in\mathcal{NP} \\ k_\epsilon \geq j}} |\epsilon| \leq D(1 + C\epsilon F(O)) \sup_{t\geq 0}(V_j^m(t) + V_j^s(t)),$$

we obtain (2). In the same way as [3], we have by the above inequalities

Proposition 5 *For given $h > 0$, there exists a threshold $\rho > 0$ so that the approximate solution constructed by the front tracking scheme satisfies*

$$\sum_{\epsilon\in\mathcal{NP}} |\epsilon| \leq h. \tag{40}$$

Let N_0 be the number of shock waves at $t = 0$. Then there exists a certain polynomial $P(\xi, \eta)$ such that

$$\begin{aligned}\sum_{\epsilon \in \mathcal{NP}} |\epsilon| &= \sum_{\substack{\epsilon \in \mathcal{NP} \\ k_\epsilon \le j}} |\epsilon| + \sum_{\substack{\epsilon \in \mathcal{NP} \\ k_\epsilon \ge j+1}} |\epsilon| \\ &= O(1)P(N_0, h^{-1})\rho + O(1)(V^m_{j+1}(t) + V^s_{j+1}(t)) \\ &= O(1)P(N_0, h^{-1})\rho + O(1)(\kappa^j + \mu_{j+1}).\end{aligned}$$

Hence, we choose j such that $O(1)(\kappa^j + \mu_{j+1}) \le \frac{h}{2}$ and then ρ so that (40) holds.

In this way, we have obtained a uniform bound of non-physical waves and hence $T.V.U^h(*, t)$. The existence of a global solution is proved by the usual argument in [3] and Smoller [11].

Theorem 4 Under the same assumption as Theorem 2, the wave-front tracking scheme is stable and provides a global in time solution.

Remark 2 *In Liu [7], global solutions are obtained provided the initial data satisfy $TV\eta_0 = H_0 < \infty$ and $(\gamma - 1)TVv_0$, $(\gamma - 1)TVu_0$, are sufficiently small (smallness depends on H_0 and equations). The above theorem does not cover this existence theorem.*

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